§11.9 Representation of Functions as Power Series

Basic Formula: $1+x+x^2+\cdots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$, $|x|<1$.

Rewrite as:

$$\star \quad \boxed{\frac{1}{1-\blacksquare} = \sum_{n=0}^{\infty} \blacksquare^n \quad \text{for } |\blacksquare|<1}$$

Goal: Want to use the above formula to REPRENT a FUNCTION as a ~~POWER SERIES~~. and FIND its RADIUS of CONVERGENCE.

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e.g. 1 (S16, 15pts) Consider the function $g(x)=\frac{5}{1-4x^2}$. (a) Express g as a power series in sigma-notation. (b). What is the radius of convergence for this series.

solution: (a). $g(x) = 5 \cdot \frac{1}{1-\boxed{4x^2}}$ apply ★ with $\blacksquare = 4x^2$

$$= 5\cdot\sum_{n=0}^{\infty}(4x^2)^n = \boxed{\sum_{n=0}^{\infty} 5\cdot(4x^2)^n} = \sum_{n=0}^{\infty} \underbrace{5\cdot 4^n}_{C_n} \cdot x^{2n}$$

(full credits at this step).

(b). The series converges if $|\blacksquare|<1$, i.e.,

$|4x^2|<1$ (keep in mind that Power Series is essentially a Geometric series which converges when $|r|<1$)

$\Rightarrow x^2<\frac{1}{4} \Rightarrow |x|<\sqrt{\frac{1}{4}} = \boxed{\frac{1}{2}}$

Therefore, the radius of convergence is $\boxed{\frac{1}{2}}$ ※.

Remarks:

①. The problem may ask you to find the first several non-zero terms. (Let's say 3 for ex),

1st non-zero term $=5$, 2nd non-zero term $=5\cdot4\cdot x^2$, 3rd $=5\cdot4^2\cdot x^4$

since $\sum_{n=0}^{\infty}5\cdot(4x^2)^n = 5\cdot(4x^2)^0 + 5\cdot(4x^2)^1 + 5\cdot(4x^2)^2 + \cdots$.

② It may also ask you "If the series is in $\sum_{n=0}^{\infty} C_n\cdot x^{2n}$ form, what is C_n?"

$\boxed{C_n = 5\cdot 4^n}$, $n=0,1,2,\cdots$. In ①, we $\boxed{\text{include } x}$ and in ② we donot.

eg 2 Represent $\frac{3x}{3+x^2}$ as a power series and find its radius of conv.
(s/b, MC).

Remark: The crucial step is to find $\boxed{1}$ and $\boxed{\blacksquare}$ in the denominator $\frac{1}{1-\blacksquare}$
(create)

$$3+x^2 = 3\cdot\left[1+\frac{x^2}{3}\right] = 3\cdot\left[1-\left(-\frac{x^2}{3}\right)\right]$$

It is important to have a negative sign here.

$\boxed{\text{Solution}}$: $\frac{3x}{3+x^2} = 3x\cdot\frac{1}{3+x^2} = 3x\cdot\frac{1}{3\left[1-\left(-\frac{x^2}{3}\right)\right]}$

$$= 3x\cdot\frac{1}{3}\cdot\frac{1}{1-\left(-\frac{x^2}{3}\right)} \qquad \text{Hint: } \blacksquare = -\frac{x^2}{3} = -\frac{1}{3}\cdot x^2$$

$$(\dagger) \quad = x\cdot\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n \qquad |\blacksquare| < 1.$$

$$= \sum_{n=0}^{\infty} x\cdot\left(-\frac{1}{3}\right)^n\cdot(x^2)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n\cdot x^{1+2n}$$

Radius of Conv: $\boxed{\left|-\frac{x^2}{3}\right|<1} \Rightarrow |x^2|<3 \Rightarrow |x|<\boxed{\sqrt{3}}$. $\boxed{R=\sqrt{3}}$

Remarks: ① If the problem only ask you to find the first several terms in the sum, then you can expand the sum at $(\dagger)$, ie.

$$x\cdot\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n = x\cdot\left[\left(-\frac{x^2}{3}\right)^0+\left(-\frac{x^2}{3}\right)^1+\left(-\frac{x^2}{3}\right)^2+\dots\right]$$

$$= x\cdot\left[1-\frac{x^2}{3}+\frac{x^4}{9}-\dots\right]$$

$$= \boxed{x-\frac{x^3}{3}+\frac{x^5}{9}-\dots}$$

② $\sqrt{x^2}=|x|$. Do not forget the absolute value when you reduce inequality $x^2<a$ to $|x|<\sqrt{a}$ and the open interval of conv is $(-\sqrt{a},\sqrt{a})$. (You will not be asked to test endpoints for Representation Prob)

③ You need to make the constant part to 1 not the x part. The following is

XXXXXXX
WRONG ATTEPT:
XXXXXXX

$$\cancel{\frac{3x}{x^2+3} = \frac{3x}{x^2\cdot\left[1+\frac{3}{x^2}\right]} = \frac{3}{x}\cdot\frac{1}{1-\left(\frac{-3}{x^2}\right)} = \frac{3}{x}\cdot\sum_{n=0}^{\infty}\left(\frac{-3}{x^2}\right)^n}$$

we need positive power of x.

eg 3. Find the power series and its radius of conv for $f(x) = \frac{x^2}{2+x}$.
(5.15)

Solution: $f(x) = x^2 \cdot \frac{1}{2+x} = x^2 \cdot \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}$

$$= \frac{x^2}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^2}{2} \cdot \frac{(-1)^n \cdot x^n}{2^n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot x^{n+2}}$$

for $\left|-\frac{x}{2}\right| < 1$, i.e. $|x| < \boxed{2}$, radius of conv $R = 2$, $c_n = \frac{(-1)^n}{2^{n+1}}$

★★★ (Advanced Topics) Differential and integration of Power Series (related to ww 6 and 7). If $f(x) = \sum_{n=0}^{\infty} c_n \cdot x^n$, then

$$f'(x) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)' = \sum_{n=0}^{\infty} (c_n \cdot x^n)' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1} \text{ and}$$

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n \cdot x^n dx = \sum_{n=0}^{\infty} \int c_n x^n dx = \sum_{n=0}^{\infty} c_n \cdot \frac{1}{n+1} \cdot x^{n+1}$$

for $|x| < 1$.

eg 4. Express $g(x) = \frac{3}{(1-x)^3}$ and find its radius of convergence as power series
(5.14).

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Take derivatives both sides: $\frac{-1}{(1-x)^2} \cdot (-1) = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n \cdot x^{n-1}$ (the first term $n=0$ is constant derivative is 0)

Take derivative again: $\left(\frac{1}{(1-x)^2}\right)' = \left(\sum_{n=1}^{\infty} n \cdot x^{n-1}\right)'$

chain rule:
left hand side: $\frac{-2}{(1-x)^3} \cdot (-1) = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot x^{n-2}$

Then $g(x) = \frac{3}{(1-x)^3} = \frac{3}{2} \cdot \boxed{\frac{2}{(1-x)^3}} = \frac{3}{2} \cdot \sum_{n=2}^{\infty} n(n-1) \cdot x^{n-2} = \sum_{n=2}^{\infty} \frac{3}{2} n(n-1) \cdot x^{n-2}$

for $|x| < 1$.

eg 5. Represent $f(x) = \tan^{-1}(7x)$ as power series and find
(ww7) its radius of convergence

Hint: $(\tan^{-1}x)' = \frac{1}{1+x^2}$ where we have representation for $\frac{1}{1+x^2}$.

Solution: $f'(x) = \frac{7}{1+(7x)^2} = \frac{7}{1-[-(7x)^2]} = 7 \cdot \sum_{n=0}^{\infty} [-(7x)^2]^n$

$$(+) \qquad = 7 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n} \cdot x^{2n}$$

Then $f(x) = \int f'(x)\,dx = \int \cancel{7} \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} x^{2n}\,dx$

$$= \sum_{n=0}^{\infty} \int (-1)^n \cdot 7^{2n+1} \cdot x^{2n}\,dx.$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} \frac{x^{2n+1}}{2n+1}$$

Radius of Conv: $|-(7x)^2| < 1 \Rightarrow (7x)^2 < 1$

$R = \frac{1}{7}$. $\qquad \Rightarrow |7x| < 1$

$$\Rightarrow \boxed{|x| < \frac{1}{7}}$$

Remark: In the webwork, you do not need to integrate the whole series (all terms). You only need to integrate the first three terms in (+), ie.

$f'(x) = 7 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n} \cdot x^{2n} = 7 \cdot (1 - 7^2 \cdot x^2 + 7^4 \cdot x^4 + \dots) = 7 - 7^3 \cdot x^2 + 7^5 \cdot x^4 + \dots$

$f(x) = \int (7 - 7^3 x^2 + 7^5 \cdot x^4 + \dots)\,dx = 7x - 7^3 \cdot \frac{1}{3} x^3 + 7^5 \cdot \frac{1}{5} \cdot x^5 + \dots$